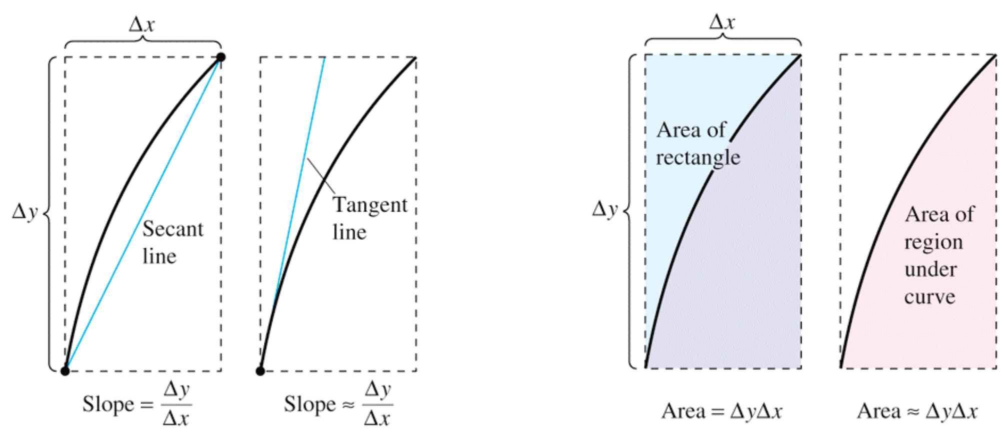
***Section 4.4** The Fundamental Theorem of Calculus*

The Fundamental Theorem of Calculus

You have now been introduced to the two major branches of calculus: differential calculus (introduced with the tangent line problem) and integral calculus (introduced with the area problem). At this point, these two problems might seem unrelated—but there is a very close connection. The connection was discovered independently by Isaac Newton and Gottfried Leibniz and is stated in a theorem that is appropriately called the **Fundamental Theorem of Calculus.**

Informally, the theorem states that differentiation and (definite) integration are inverse operations, in the same sense that division and multiplication are inverse operations. To see how Newton and Leibniz might have anticipated this relationship, consider the approximations shown in Figure 4.26. The slope of the tangent line was defined using the *quotient* $\Delta y/\Delta x$ (the slope of the secant line). Similarly, the area of a region under a curve was defined using the *product* $\Delta y \Delta x$ (the area of a rectangle). So, at least in the primitive approximation stage, the operations of differentiation and definite integration appear to have an inverse relationship in the same sense that division and multiplication are inverse operations. The Fundamental Theorem of Calculus states that the limit processes (used to define the derivative and definite integral) preserve this inverse relationship.



(a) Differentiation **(b)** Definite integration

Differentiation and definite integration have an "inverse" relationship.

Figure 4.26

THEOREM 4.9 The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

PROOF The key to the proof is in writing the difference $F(b) - F(a)$ in a convenient form. Let Δ be any partition of $[a, b]$.

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_{n-1} < x_n = b$$

By pairwise subtraction and addition of like terms, you can write

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_{n-1}) + F(x_{n-1}) - \cdot \cdot \cdot - F(x_1) + F(x_1) - F(x_0) \\ &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]. \end{aligned}$$

By the Mean Value Theorem, you know that there exists a number c_i in the ith subinterval such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}.$$

Because $F'(c_i) = f(c_i)$, you can let $\Delta x_i = x_i - x_{i-1}$ and obtain

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\,\Delta x_i.$$

This important equation tells you that by repeatedly applying the Mean Value Theorem, you can always find a collection of c_i's such that the *constant* $F(b) - F(a)$ is a Riemann sum of f on $[a, b]$ for any partition. Theorem 4.4 guarantees that the limit of Riemann sums over the partition with $\|\Delta\| \to 0$ exists. So, taking the limit (as $\|\Delta\| \to 0$) produces

$$F(b) - F(a) = \int_a^b f(x)\,dx. \qquad \blacksquare$$

The following guidelines can help you understand the use of the Fundamental Theorem of Calculus.

Guidelines for Using the Fundamental Theorem of Calculus

1. *Provided you can find* an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus, the following notation is convenient.

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b$$
$$= F(b) - F(a)$$

For instance, to evaluate $\int_1^3 x^3\,dx$, you can write

$$\int_1^3 x^3\,dx = \frac{x^4}{4}\Big]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

3. It is not necessary to include a constant of integration C in the antiderivative because

$$\int_a^b f(x)\,dx = \Big[F(x) + C\Big]_a^b$$
$$= [F(b) + C] - [F(a) + C]$$
$$= F(b) - F(a).$$

Ex.1 Evaluating a Definite Integral

Evaluate each definite integral.

a. $\int_1^7 (6x^2 + 2x - 3)\,dx$

b. $\displaystyle\int_{-8}^{-1} \frac{x - x^2}{2\sqrt[3]{x}}\, dx$

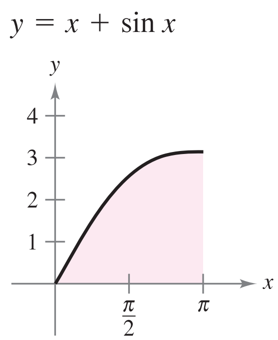
c. $\displaystyle\int_{\pi/4}^{\pi/2} (2 - \csc^2 x)\, dx$

Ex.2 Evaluating a Definite Integral Involving Absolute Value

Evaluate $\int_1^4 (3 - |x - 3|)\, dx$

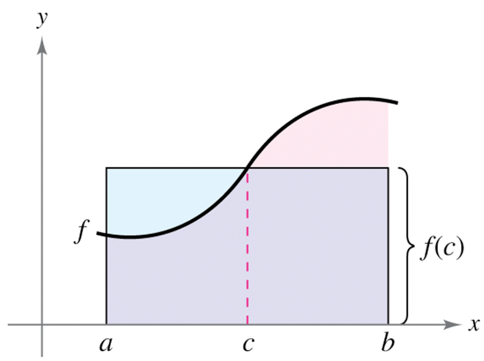
Ex.3 Using the Fundamental Theorem to Find Area

Find the area of the region bounded by the graph of $y = x + \sin(x)$, the x-axis, and the vertical lines $x = 0$ and $x = \pi$.



The Mean Value Theorem for Integrals

In Section 4.2, you saw that the area of a region under a curve is greater than the area of an inscribed rectangle and less than the area of a circumscribed rectangle. The Mean Value Theorem for Integrals states that somewhere "between" the inscribed and circumscribed rectangles there is a rectangle whose area is precisely equal to the area of the region under the curve, as shown in Figure 4.29.



Mean value rectangle:

$$f(c)(b-a) = \int_a^b f(x)\,dx$$

Figure 4.29

THEOREM 4.10 Mean Value Theorem for Integrals

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

NOTE Notice that Theorem 4.10 does not specify how to determine c. It merely guarantees the existence of at least one number c in the interval. ■

Ex.4 Find the value(s) of ***c*** guaranteed by the Mean Value Theorem for Integrals for $f(x) = \cos(x)$ over the interval $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$.

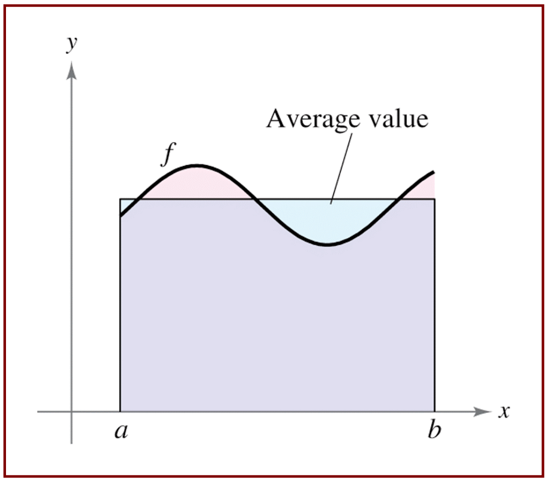
Average Value of a Function

The value of $f(c)$ given in the Mean Value Theorem for Integrals is called the **average value** of f on the interval $[a, b]$.

Definition of the Average Value of a Function on an Interval

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$



$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Figure 4.31

NOTE Notice in Figure 4.31 that the area of the region under the graph of f is equal to the area of the rectangle whose height is the average value.

To see why the average value of f is defined in this way, suppose that you partition $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. If c_i is any point in the ith subinterval, the arithmetic average (or mean) of the function values at the c_i's is given by

$$a_n = \frac{1}{n}[f(c_1) + f(c_2) + \cdot\cdot\cdot + f(c_n)]. \qquad \text{Average of } f(c_1), \ldots, f(c_n)$$

By multiplying and dividing by $(b - a)$, you can write the average as

$$\begin{aligned} a_n &= \frac{1}{n}\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{b-a}\right) = \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{n}\right) \\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\,\Delta x. \end{aligned}$$

Finally, taking the limit as $n \to \infty$ produces the average value of f on the interval $[a, b]$, as given in the definition above.

Ex.5 Finding the Average Value of a Function

Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$.

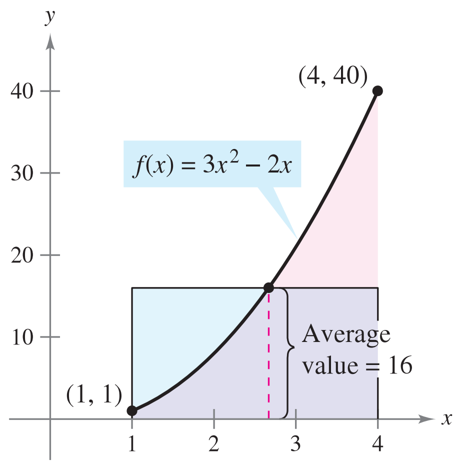


Figure 4.32

$$\frac{1}{b-a}\int_a^b f(x)\,dx =$$

The Second Fundamental Theorem of Calculus

Earlier you saw that the definite integral of f on the interval $[a, b]$ was defined using the constant b as the upper limit of integration and x as the variable of integration. However, a slightly different situation may arise in which the variable x is used in the upper limit of integration. To avoid the confusion of using x in two different ways, t is temporarily used as the variable of integration. (Remember that the definite integral is *not* a function of its variable of integration.)

The Definite Integral as a Number

Constant

$$\int_a^b f(x)\,dx$$

Constant

f is a function of x.

The Definite Integral as a Function of x

F is a function of x.

$$F(x) = \int_a^x f(t)\,dt$$

Constant

f is a function of t.

Ex.6 The Definite Integral as a Function

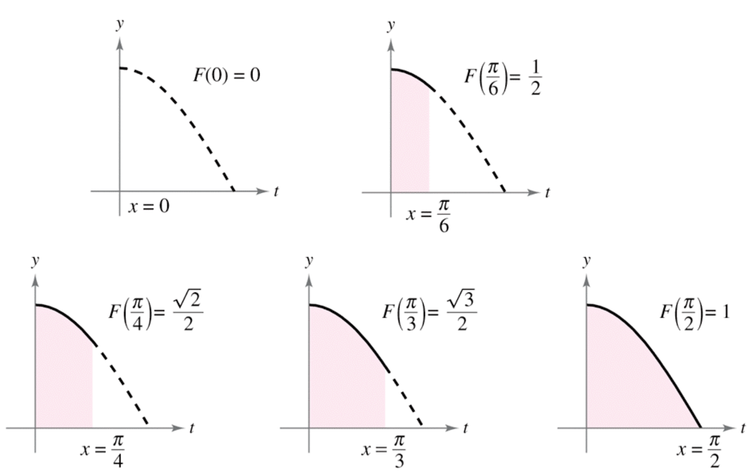
Evaluate the function

$$F(x) = \int_0^x \cos t\,dt$$

at $x = 0, \pi/6, \pi/4, \pi/3$, and $\pi/2$.

$$\int_0^x \cos t\,dt = \sin t\Big]_0^x = \sin x - \sin 0 = \sin x.$$

Now, using $F(x) = \sin x$, you can obtain the results shown in Figure 4.34.



$F(x) = \int_0^x \cos t\,dt$ is the area under the curve $f(t) = \cos t$ from 0 to x.

$F(x) = \int_0^x \cos t\,dt$ is the area under the curve $f(t) = \cos t$ from 0 to x.

Figure 4.34

You can think of the function $F(x)$ as *accumulating* the area under the curve $f(t) = \cos t$ from $t = 0$ to $t = x$. For $x = 0$, the area is 0 and $F(0) = 0$. For $x = \pi/2$, $F(\pi/2) = 1$ gives the accumulated area under the cosine curve on the entire interval $[0, \pi/2]$. This interpretation of an integral as an **accumulation function** is used often in applications of integration.

In Example 6, note that the derivative of F is the original integrand (with only the variable changed). That is,

$$\frac{d}{dx}[F(x)] = \frac{d}{dx}[\sin x] = \frac{d}{dx}\left[\int_0^x \cos t\, dt\right] = \cos x.$$

This result is generalized in the following theorem, called the **Second Fundamental Theorem of Calculus.**

THEOREM 4.11 The Second Fundamental Theorem of Calculus

If f is continuous on an open interval I containing a, then, for every x in the interval,

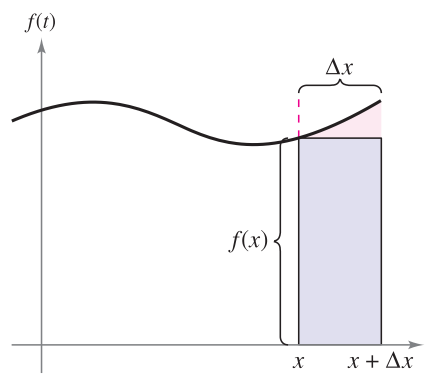
$$\frac{d}{dx}\left[\int_a^x f(t)\, dt\right] = f(x).$$

PROOF Begin by defining F as

$$F(x) = \int_a^x f(t)\, dt.$$

Then, by the definition of the derivative, you can write

$$\begin{aligned} F'(x) &= \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_a^{x+\Delta x} f(t)\, dt - \int_a^x f(t)\, dt\right] \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_a^{x+\Delta x} f(t)\, dt + \int_x^a f(t)\, dt\right] \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_x^{x+\Delta x} f(t)\, dt\right]. \end{aligned}$$



$$f(x)\,\Delta x \approx \int_x^{x+\Delta x} f(t)\, dt$$

Figure 4.35

From the Mean Value Theorem for Integrals (assuming $\Delta x > 0$), you know there exists a number c in the interval $[x, x + \Delta x]$ such that the integral in the expression above is equal to $f(c)\,\Delta x$. Moreover, because $x \le c \le x + \Delta x$, it follows that $c \to x$ as $\Delta x \to 0$. So, you obtain

$$\begin{aligned} F'(x) &= \lim_{\Delta x \to 0}\left[\frac{1}{\Delta x} f(c)\,\Delta x\right] \\ &= \lim_{\Delta x \to 0} f(c) \\ &= f(x). \end{aligned}$$

A similar argument can be made for $\Delta x < 0$. ■

NOTE Using the area model for definite integrals, you can view the approximation

$$f(x)\,\Delta x \approx \int_x^{x+\Delta x} f(t)\,dt$$

as saying that the area of the rectangle of height $f(x)$ and width Δx is approximately equal to the area of the region lying between the graph of f and the x-axis on the interval $[x, x + \Delta x]$, as shown in Figure 4.35.

Note that the Second Fundamental Theorem of Calculus tells you that if a function is continuous, you can be sure that it has an antiderivative. This antiderivative need not, however, be an elementary function. (Recall the discussion of elementary functions in Section P.3.)

Ex.7 Using the Second Fundamental Theorem of Calculus

(a) Integrate to find $F(x)$. **(b)** Then, demonstrate the Second Fundamental Theorem of Calculus to find $F'(x)$ by differentiating your result in part (a).

$$F(x) = \int_0^x t(t^2 + 1)\,dt$$

Ex.8 Using the Second Fundamental Theorem of Calculus

Find the derivative of $F(x) = \int_2^{x^2} \frac{1}{t^3}\, dt$

Using $u =$

$$F'(x) = \frac{dF}{du}\frac{du}{dx}$$

$$= \frac{d}{du}[F(x)]\frac{du}{dx}$$

Net Change Theorem

The Fundamental Theorem of Calculus (Theorem 4.9) states that if f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

But because $F'(x) = f(x)$, this statement can be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

where the quantity $F(b) - F(a)$ represents the *net change of F* on the interval $[a, b]$.

THEOREM 4.12 THE NET CHANGE THEOREM

The definite integral of the rate of change of a quantity $F'(x)$ gives the total change, or **net change,** in that quantity on the interval $[a, b]$.

$$\int_a^b F'(x)\,dx = F(b) - F(a) \qquad \text{Net change of } F$$

Ex.9 Using the Net Change Theorem

A chemical flows into a storage tank at a rate of $180 + 3t$ liters per minute, where $0 \le t \le 60$. Find the amount of the chemical that flows into the tank during the first 20 minutes.

Another way to illustrate the Net Change Theorem is to examine the velocity of a particle moving along a straight line where $s(t)$ is the position at time t. Then its velocity is $v(t) = s'(t)$ and

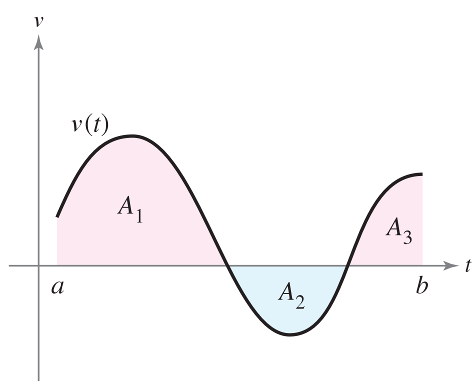
$$\int_a^b v(t)\,dt = s(b) - s(a).$$

This definite integral represents the net change in position, or **displacement,** of the particle.

When calculating the *total* distance traveled by the particle, you must consider the intervals where $v(t) \le 0$ and the intervals where $v(t) \ge 0$. When $v(t) \le 0$, the particle moves to the left, and when $v(t) \ge 0$, the particle moves to the right. To calculate the total distance traveled, integrate the absolute value of velocity $|v(t)|$. So, the displacement of a particle and the total distance traveled by a particle over $[a, b]$ can be written as

Displacement on $[a, b]$ $= \int_a^b v(t)\,dt = A_1 - A_2 + A_3$

Total distance traveled on $[a, b]$ $= \int_a^b |v(t)|\,dt = A_1 + A_2 + A_3$



A_1, A_2, and A_3 are the areas of the shaded regions.
Figure 4.36

Ex.10 Solving a Particle Motion Problem

A particle is moving along a line so that its velocity is $v(t) = t^3 - 8t^2 + 15t$ feet per second at time t.

(a) What is the displacement of the particle on the time interval $0 \le t \le 5$?

(b) What is the total distance traveled by the particle on the time interval $0 \le t \le 5$?